since rotations in the (1-2) plane and in the (3-4) plane are like gangsters operating on different turfs. Next, we tackle $[J_{(23)}, J_{(31)}]$. Notice that the action takes place entirely in the $SO(3)$ subgroup of $SO(4)$, and so we already know the answer: $[J_{(23)}, J_{(31)}] = [J_x, J_y] = iJ_z = iJ_{(12)}$. These two examples, together with antisymmetry $J_{(mn)} = -J_{(nm)}$, in fact take care of all possible cases. In the commutator $[J_{(mn)}, J_{(pq)}]$, there are three possibilities for the index sets (mn) and (pq): (i) they have no integer in common, (ii) they have one integer in common, or (iii) they have two integers in common. The commutator vanishes in cases (i) and (iii), for trivial (but different) reasons. In case (ii), suppose $m = p$ with no loss of generality, then the commutator is equal to $iJ_{(nq)}$.

We obtain, for any N,

$$[J_{(mn)}, J_{(pq)}] = i(\delta_{mp}J_{(nq)} + \delta_{nq}J_{(mp)} - \delta_{np}J_{(mq)} - \delta_{mq}J_{(np)}) \tag{10.26}$$

This may look rather involved to the uninitiated, but in fact it simply states in mathematical symbols the last three sentences of the preceding paragraph. First, on the right-hand side, a linear combination of the Js (as required by the general argument above) is completely fixed by the first term by noting that the left-hand side is antisymmetric under three separate interchanges: $m \leftrightarrow n$, $p \leftrightarrow q$, and $(mn) \leftrightarrow (pq)$. Next, all those Kronecker deltas just say that if the two sets (mn) and (pq) have no integer in common, then the commutator vanishes. If they do have an integer in common, simply "cross off" that integer. For example, $[J_{(12)}, J_{(14)}] = iJ_{(24)}$ and $[J_{(23)}, J_{(31)}] = -iJ_{(21)} = iJ_{(12)}$.

10.2 Lie Algebra of SO(3) and Ladder Operators: Creation and Annihilation (A trimmed copy of (IV.2 of GTNFP))

In this section we will consider higher dimensional representations of S0(3) and then look into how to find its irreducible representations. This should, similarly to the previous section, feel very familiar. You were essentially already shown how to do this when you were first introduced to quantum angular momentum! However, walking through this carefully will give us the tools we need in the next section to tackle the irreducible representations of tensor product reps of SO(3) more carefully (i.e., re-study the additional of angular momentum).

10.2.1 Ladder operators are useful (a recap of stuff you've seen before)

Since the three generators J_x, J_y, and J_z do not commute, they cannot be simultaneously diagonalized, as explained in the review of linear algebra. But we can diagonalize one of them. Choose J_z, and work in a basis in which J_z is diagonal.

The move that breaks the problem wide open should be very familiar to you: it is akin to going from the 2-dimensional coordinates x, y to the complex variable $z = x + iy, z^* = x - iy$, and from a transversely polarized electromagnetic wave to a circularly polarized electromagnetic wave. Define $J_\pm \equiv J_x \pm iJ_y$. Then we can rewrite (10.20) as

$$[J_z, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_z. \tag{10.27}$$

Write the eigenvector of J_z with eigenvalue m as $|m\rangle$; in other words,

$$J_z|m\rangle = m|m\rangle. \tag{10.28}$$

Since J_z is hermitean, m is a real number. What we are doing is going to a basis in which J_z is diagonal; according to (10.27), $J_\pm$ cannot be diagonal in this basis. Now consider the state $J_+|m\rangle$ and act on it with J_z:

$$J_z J_+|m\rangle = (J_+J_z + [J_z, J_+])|m\rangle = (J_+J_z + J_+)|m\rangle = (m+1)J_+|m\rangle, \tag{10.29}$$

where the second equality follows from (10.27). (Henceforth, we will be using (10.27) repeatedly without bothering to refer to it.)

Thus, $J_+|m\rangle$ is an eigenvector (or eigenstate; these terms are used interchangeably) of J_z with eigenvalue $m+1$. Hence, by the definition of $|m\rangle$, the state $J_+|m\rangle$ must be equal to the state $|m+1\rangle$ multiplied by some normalization constant; in other words, we have

$$J_+|m\rangle = c_{m+1}|m+1\rangle, \tag{10.30}$$

with the complex number c_{m+1} to be determined. Similarly,

$$J_z J_-|m\rangle = (J_-J_z + [J_z, J_-])|m\rangle = (J_-J_z - J_-)|m\rangle = (m-1)J_-|m\rangle, \tag{10.31}$$

from which we conclude that

$$J_-|m\rangle = b_{m-1}|m-1\rangle, \tag{10.32}$$

with some other unknown normalization constant.

It is very helpful to think of the states $\cdots, |m-1\rangle, |m\rangle, |m+1\rangle, \cdots$ as corresponding to rungs on a ladder. The result $J_+|m\rangle = c_{m+1}|m+1\rangle$ tells us that we can think of J_+ as a "raising operator" that enables us to climb up one rung on the ladder, going from $|m\rangle$ to $|m+1\rangle$. Similarly, the result $J_-|m\rangle = b_{m-1}|m-1\rangle$ tells us to think of J_- as a "lowering operator" that enables us to climb down one rung on the ladder. Collectively, $J_\pm$ are referred to as ladder operators.

To relate b_m to c_m, we invoke the hermiticity of J_x, J_y, and J_z, which implies that

$$(J_+)^\dagger = (J_x + iJ_y)^\dagger = J_x - iJ_y = J_-.$$

Multiplying $J_+|m\rangle = c_{m+1}|m+1\rangle$ from the left by $\langle m+1|$ and normalizing the states by $\langle m|m\rangle = 1$, we obtain

$$\langle m+1|J_+|m\rangle = c_{m+1}.$$

Complex conjugating this gives us $c^*_{m+1} = \langle m|J_-|m+1\rangle = b_m$, that is, $b_{m-1} = c^*_m$, so that we can write

$$J_-|m\rangle = c^*_m|m-1\rangle.$$

Acting on this with J_+ gives

$$J_+J_-|m\rangle = c^*_m J_+|m-1\rangle = |c_m|^2|m\rangle.$$

Similarly, acting with J_-J_+ on $|m\rangle$ gives

$$J_-J_+|m\rangle = c_{m+1}|m+1\rangle \implies |c_{m+1}|^2|m\rangle.$$

Since we know that the representation is finite dimensional, the ladder must terminate, that is, there must be a top rung. So, call the maximum value of m by j. At this stage, all we know is that j is a real number. (Note that we have not assumed that m is an integer.) Thus, there is a state $|j\rangle$ such that $J_+|j\rangle = 0$. It corresponds to the top rung of the ladder.

At this point, we have only used the first part of (10.27). Now we use the second half:

$$\langle j|J_+J_-|j\rangle = \langle j|J_-J_+ - 2J_z|j\rangle = |c_j|^2 - 2j,$$

thus determining $|c_j|^2 = 2j$. Furthermore,

$$\langle m|J_+J_-|m\rangle = \langle m|(J_+J_- - J_-J_+)|m\rangle = |c_m|^2 - |c_{m+1}|^2 = 2m.$$

We obtain a recursion relation

$$|c_m|^2 = |c_{m+1}|^2 + 2m,$$

which, together with $|c_j|^2 = 2j$, allows us to determine the unknown $|c_m|$. Here we go:

$$|c_{j-1}|^2 = |c_j|^2 + 2(j-1) = 2(2j-1),$$

then

$$|c_{j-2}|^2 = |c_{j-1}|^2 + 2(j-2) = 2(3j-1-2),$$

and eventually

$$|c_{j-s}|^2 = 2\big((s+1)j - \sum_{i=1}^{s} i\big).$$

Recall the Gauss formula $\sum_{i=1}^{s} i = \frac{1}{2}s(s+1)$, and obtain

$$|c_{j-s}|^2 = 2\big((s+1)j - \frac{1}{2}s(s+1)\big) = (s+1)(2j-s).$$

We keep climbing down the ladder, increasing s by 1 at each step. When $s = 2j$, we see that c_{-j} vanishes. We have reached the bottom of the ladder. More explicitly, we have

$$J_-|-j\rangle = c_{-j}^*|-j-1\rangle = 0,$$

according to what we just derived. The minimum value of m is $-j$. Since s counts the number of rungs climbed down, it is necessarily an integer, and thus the condition $s = 2j$ that the ladder terminates implies that j is either an integer or a half-integer, depending on whether s is even or odd. If the ladder terminates, then we have the set of states $|-j\rangle, |-j+1\rangle, \ldots, |j-1\rangle, |j\rangle$, which totals $2j+1$ states.

For example, for $j = 2$, these states are $|-2\rangle, |-1\rangle, |0\rangle, |1\rangle, |2\rangle$. Starting from $|2\rangle$, we apply J_- four times to reach $|-2\rangle$. (We will do this explicitly later in this chapter.) To emphasize the dependence on j, we sometimes write the kets $|m\rangle$ as $|j,m\rangle$. Notice that the ladder is symmetric under $|m\rangle \to |-m\rangle$, a symmetry that can be traced to the invariance of the algebra in (10.20) under $J_x \to J_x, J_y \to -J_y$, and $J_z \to -J_z$ (namely, a rotation through π around the x-axis).

Mysterious Appearance of the Half-Integers. But what about the representations of the algebra corresponding to j = a half-integer? For example, for $j = \frac{1}{2}$, we have a $2 \cdot \frac{1}{2} + 1 = 2$-dimensional representation consisting of the states $\left|-\frac{1}{2}\right\rangle$ and $\left|\frac{1}{2}\right\rangle$. We climb down from $\left|\frac{1}{2}\right\rangle$ to $\left|-\frac{1}{2}\right\rangle$ in one step. Certainly no sight of a 2-dimensional representation in chapter I.3! The mystery of the $j = \frac{1}{2}$ representation will be resolved in chapter IV.5 when we discuss $SU(2)$, but let's not be coy about it and keep the reader in suspense. I trust that most readers have heard that it describes the electron spin. We did not go looking for the peculiar number, it came looking for us.

It should not escape your notice that as a by-product of requiring the ladder to terminate, we have also determined $|c_m|^2$. Indeed, setting $s = j - m$, we had $|c_m|^2 = (j+m)(j-m+1)$. Recalling the definition of c_m, we obtain

$$J_+|m\rangle = c_{m+1}|m+1\rangle = \sqrt{(j+1+m)(j-m)}|m+1\rangle. \tag{10.33}$$

and

$$J_-|m\rangle = c_m^*|m-1\rangle = \sqrt{(j+1-m)(j+m)}|m-1\rangle. \tag{10.34}$$

As a mild check on the arithmetic, indeed $J_+|j\rangle = 0$ and $J_-|-j\rangle = 0$. You might also have noticed that, quite rightly, the phase of c_m is not determined, since it is completely up to us to choose the relative phase of the kets $|m\rangle$ and $|m-1\rangle$. Beware that different authors choose differently. I simply take c_m to be real and positive. Tables of the c_ms for various js are available, but it's easy enough to write them down when needed. Note also that the square roots in (10.33) and (10.34) are related by $m \leftrightarrow -m$.

Example of ladder operators. For convenience, let's list here the two most common cases needed in physics. For $j = \frac{1}{2}$:

$$J_+\left|-\frac{1}{2}\right\rangle = \left|\frac{1}{2}\right\rangle, \quad J_-\left|\frac{1}{2}\right\rangle = \left|-\frac{1}{2}\right\rangle. \tag{10.35}$$

For $j = 1$:

$$J_+|-1\rangle = \sqrt{2}|0\rangle, \quad J_+|0\rangle = \sqrt{2}|1\rangle, \quad J_-|1\rangle = \sqrt{2}|0\rangle, \quad J_-|0\rangle = \sqrt{2}|-1\rangle. \tag{10.36}$$

Note that the (nonzero) c_m for these two cases are particularly easy to remember (that is, if for some odd reason you want to): they are all 1 in one case, and $\sqrt{2}$ in the other. Let us also write down the $j = 2$ case for later use:

$$\begin{gathered} J_+|-2\rangle = \sqrt{2}|-1\rangle, \quad J_+|-1\rangle = \sqrt{6}|0\rangle, \quad J_+|0\rangle = \sqrt{6}|1\rangle, \quad J_+|1\rangle = \sqrt{2}|2\rangle, \\ J_-|2\rangle = \sqrt{2}|1\rangle, \quad J_-|1\rangle = \sqrt{6}|0\rangle, \quad J_-|0\rangle = \sqrt{6}|-1\rangle, \quad J_-|-1\rangle = \sqrt{2}|-2\rangle. \end{gathered} \tag{10.37}$$

So you did all of this before in QP1 and might be wondering what is new so what have you learnt from this? We'll we've implicitly figured out how to write J_+ and J_-, and thereby also J_+ and J_- in a $2j+1$ dimensional basis working only from the known commutation relationships between J_x, J_y and J_z. Or, in group theoretic language, from the structure constants that define the Lie Algebra of 3D rotations, $SO(3)$, we have computed a $2j+1$ dimensional representation of the $SO(3)$ Lie algebra.

10.3 Addition of Angular Momentum (e.g. multiplying SO(3) representations)

This is Section IV.3, pg. 217 of GTNFP. Note both here and earlier I have chosen to skip constructing high dimensional representations, and finding their irreps, via tensors in favour of a more familiar ladder operator (i.e., lie algebraic) approach. But if you're interested and have time do go and read those bits from GTNFP.

In the prototypical quantum mechanical problem, two particles orbit in a spherically symmetric potential. Particle unprime could be in the state $|l, m\rangle$, and particle prime in the state $|l', m'\rangle$. If the particles do not interact, then the eigenstates of the Hamiltonian could be written using the product states $|l,m\rangle \otimes |l',m'\rangle$. But the particles do interact with each other, and the Hamiltonian H then includes an interaction term H_I (which we take to depend only on the distance between the two particles). To leave H invariant, we would have to rotate both particles, of course. We want to understand what group theory tells us about the wave function of the two particles.

But the mathematical problem involved corresponds to breaking the tensor product of two representations of SO(3) down into its irreducible representations. Suppose we are given two irreducible representations of the Lie algebra of $SO(3)$, labeled by j and j'. We have two sets of kets: $|j,m\rangle$ with $m = -j, -j+1, \cdots, j-1, j$, and $|j',m'\rangle$ with $m' = -j', -j'+1, \cdots, j'-1, j'$. The $2j+1$ kets $|j,m\rangle$, when acted on by the generators J_i, transform into linear combinations of one another. Similarly, the $2j'+1$ kets $|j',m'\rangle$, when acted on by the generators J_i, transform into linear combinations of one another. Now we write down the product kets $|j,m\rangle \otimes |j',m'\rangle$. There are $(2j+1)(2j'+1)$ such states. When acted on by the generators J_i, these kets naturally transform into linear combinations of one another, thus furnishing a $(2j+1)(2j'+1)$-dimensional representation of $SO(3)$. We expect this representation to be reducible.

The concept of irreducibility transfers naturally from representations of a Lie group to the representations of a Lie algebra. If the matrices representing the J_is could be block diagonalized, we say that the representation is reducible. When the generators J_i act on the product kets $|j,m\rangle \otimes |j',m'\rangle$, they act on $|j,m\rangle$ and then on $|j',m'\rangle$. We can verify this more-or-less self-evident fact by rotating the product kets. Under an infinitesimal rotation around the z-axis, $R \simeq I + i\theta J_z$, both $|j,m\rangle$ and $|j',m'\rangle$ rotate, of course. Thus,

$$\begin{aligned}
|j,m\rangle \otimes |j',m'\rangle &\to R|j,m\rangle \otimes R|j',m'\rangle \\
&\simeq (I + i\theta J_z)|j,m\rangle \otimes (I + i\theta J_z)|j',m'\rangle \\
&= (I + i\theta m)|j,m\rangle \otimes (I + i\theta m')|j',m'\rangle \\
&\simeq (I + i\theta(m+m'))|j,m\rangle \otimes |j',m'\rangle + \mathcal{O}(\theta^2).
\end{aligned}$$

In other words,

$$J_z(|j,m\rangle \otimes |j',m'\rangle) = (J_z|j,m\rangle) \otimes |j',m'\rangle + |j,m\rangle \otimes (J_z|j',m'\rangle), \tag{10.38}$$

or equivalently,

$$J_z|j,m\rangle \otimes |j',m'\rangle = (m+m')|j,m\rangle \otimes |j',m'\rangle. \tag{10.39}$$

The operator J_z acts in turn on $|j,m\rangle$ and $|j',m'\rangle$. Thus, $|j,m\rangle \otimes |j',m'\rangle$ is an eigenstate of J_z with eigenvalue $m+m'$. The eigenvalues of J_z simply add.

To avoid writing $\otimes$ constantly, we denote $|j,m\rangle \otimes |j',m'\rangle$ by $|j,j',m,m'\rangle$. We just learned that $|j,j',m,m'\rangle$ is an eigenstate of J_z with eigenvalue $m+m'$. We know that the maximum values m and m' can attain are j and j', respectively, and thus the maximum eigenvalue J_z can have is $j+j'$, attained with the state $|j,j',j,j'\rangle$.

10.3.1 The Clebsch-Gordan decomposition (examinable!)

The plan of attack is to apply the lowering operator J_- repeatedly on $|j,j',j,j'\rangle$. To see what is going on, let's go through some examples.

Example (A): $j = \frac{1}{2}, j' = \frac{1}{2}$

There are $(2j+1)(2j'+1) = 2 \cdot 2 = 4$ states $\left|\frac{1}{2},\frac{1}{2},m,m'\right\rangle$ with $m = -\frac{1}{2},\frac{1}{2}$ and $m' = -\frac{1}{2},\frac{1}{2}$. Since j and j' are fixed in this discussion, we might as well omit them and simply write $|m,m'\rangle$ instead of $|j, j', m, m'\rangle$. Let's go slow and list the four states:

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle, \left|\frac{1}{2}, -\frac{1}{2}\right\rangle, \left|-\frac{1}{2}, \frac{1}{2}\right\rangle, \left|-\frac{1}{2}, -\frac{1}{2}\right\rangle.$$

As explained above, we expect these four states to furnish a reducible representation and thus to fall apart into a bunch of irreducible representations labeled by J. Let us denote the states in these irreducible representations by $|J, M\rangle$ with $M = -J, -J+1, \ldots, J$.

Of these four states, $\left|\frac{1}{2}, \frac{1}{2}\right\rangle$ has the maximum eigenvalue J_z can have, namely, $\frac{1}{2} + \frac{1}{2} = 1$. Thus, it can belong only to an irreducible representation labeled by J with $J \geq 1$. In fact, it cannot be that $J > 1$, because then there would have to be states with eigenvalue of J_z greater than 1. So we have

$$|1, 1\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle. \tag{10.40}$$

The strategy is to climb down the ladder by applying J_- repeatedly. So, act with J_- on $|1, 1\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle$. But we know from chapter IV.2 how J_- acts on these states. Using Eq. (10.36), we have

$$J_-|1, 1\rangle = \sqrt{2}|1, 0\rangle, \tag{10.41}$$

while using Eq. (10.35), we have

$$J_-\left|\frac{1}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|-\frac{1}{2}, \frac{1}{2}\right\rangle + \left|\frac{1}{2}, -\frac{1}{2}\right\rangle\right). \tag{10.42}$$

Thus,

$$|1, 0\rangle = \frac{1}{\sqrt{2}}\left(\left|-\frac{1}{2}, \frac{1}{2}\right\rangle + \left|\frac{1}{2}, -\frac{1}{2}\right\rangle\right). \tag{10.43}$$

Applying J_- again, we obtain

$$\sqrt{2}|1, -1\rangle = \frac{1}{\sqrt{2}}\left(2\left|-\frac{1}{2}, -\frac{1}{2}\right\rangle\right), \tag{10.44}$$

and thus

$$|1, -1\rangle = \left|-\frac{1}{2}, -\frac{1}{2}\right\rangle, \tag{10.45}$$

which we might have expected by applying symmetry to our starting equation, flipping the z-axis.

We have now accounted for three of the four states we started with. The only orthogonal state left is the linear combination

$$\frac{1}{\sqrt{2}}\left(\left|-\frac{1}{2}, \frac{1}{2}\right\rangle - \left|\frac{1}{2}, -\frac{1}{2}\right\rangle\right), \tag{10.46}$$

which has eigenvalue 0 under J_z; this state, all by its lonesome self, must be

$$|J = 0, M = 0\rangle. \tag{10.47}$$

Let me summarize our results, giving $|J, M\rangle$ in terms of $|m, m'\rangle$:

$$|1, 1\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle, \tag{10.48}$$

$$|1, 0\rangle = \frac{1}{\sqrt{2}}\left(\left|-\frac{1}{2}, \frac{1}{2}\right\rangle + \left|\frac{1}{2}, -\frac{1}{2}\right\rangle\right), \tag{10.49}$$

$$|1,-1\rangle = \left|-\frac{1}{2},-\frac{1}{2}\right\rangle, \tag{10.50}$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(\left|-\frac{1}{2},\frac{1}{2}\right\rangle - \left|\frac{1}{2},-\frac{1}{2}\right\rangle\right). \tag{10.51}$$

Or, in more group theoretic language, we have just shown that

$$\frac{1}{2}\otimes\frac{1}{2} = 1\oplus 0. \tag{10.52}$$

Example B:

Now that you have gone through example (A), we can practically race through this example. Start with $3\cdot 3 = 9$ states $|1,1,m,m'\rangle$ with $m=-1,0,1$ and $m'=-1,0,1$. Again, we write $|m,m'\rangle$ instead of $|j,j',m,m'\rangle$. These nine states furnish a reducible representation which decomposes into a bunch of irreducible representations labeled by J. In these irreducible representations, the states are denoted by $|J,M\rangle$ with $M=-J,-J+1,\ldots,J$.

Of these nine states, the one with the highest value of M is $|1,1\rangle$, for which $M=1+1=2$. So start with

$$|2,2\rangle = |1,1\rangle. \tag{10.53}$$

and climb down the ladder. Act with J_-, using Eq. (10.37). But as remarked in connection with example (A), we don't even need to look these up. Remembering that $|1,1\rangle$ means $|1\rangle\otimes|1\rangle$, we lower each of the two kets in turn to $|0\rangle$, so that we end up with a linear combination of $|1,0\rangle$ and $|0,1\rangle$. But by the principle of democracy, these two kets must appear with equal weight, and thus

$$|2,1\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,1\rangle\right). \tag{10.54}$$

Onward! Apply J_- again. Advocating democracy is not enough anymore, since this only tells us that we get a state proportional to $|-1,1\rangle + c|0,0\rangle + |1,-1\rangle$ with an unknown constant c. We have to invoke Eq. (10.37) to determine $c=2$. Thus,

$$|2,0\rangle = \frac{1}{\sqrt{6}}\left(|-1,1\rangle + 2|0,0\rangle + |1,-1\rangle\right). \tag{10.55}$$

At this point we could keep going, but there is no need to even apply J_- anymore. By reflection symmetry along the z-axis, we have

$$|2,-1\rangle = \frac{1}{\sqrt{2}}\left(|0,-1\rangle + |-1,0\rangle\right), \tag{10.56}$$

and

$$|2,-2\rangle = |-1,-1\rangle. \tag{10.57}$$

These account for five out of the nine states. Of the remaining states, the maximum value M can have is 1, attained by the states $|0,1\rangle$ and $|1,0\rangle$. But this state $|J=1,M=1\rangle$ has to be orthogonal to the state

$$|2,1\rangle = \frac{1}{\sqrt{2}}\left(|0,1\rangle + |1,0\rangle\right) \tag{10.58}$$

we already have. Thus, with essentially no work, we have found

$$|1,1\rangle = \frac{1}{\sqrt{2}}\left(|0,1\rangle - |1,0\rangle\right). \tag{10.59}$$

Again, apply J_- on this, and by democracy, we obtain with no work at all

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|-1,1\rangle - |1,-1\rangle\right), \tag{10.60}$$

and then

$$|1,-1\rangle = \frac{1}{\sqrt{2}}\left(|-1,0\rangle - |0,-1\rangle\right). \tag{10.61}$$

So now there is only $9-5-3=1$ state left. This lone state is determined by the fact that it is orthogonal to everybody else. Hence,

$$|0,0\rangle = \frac{1}{\sqrt{3}}\left(|-1,-1\rangle - |0,0\rangle + |1,1\rangle\right). \tag{10.62}$$

Or, in more group theoretic language, have just shown that

$$1 \otimes 1 = 2 \oplus 1 \oplus 0. \tag{10.63}$$

General case

This procedure breaking down $j \otimes j'$ into a direct sum of irreducible representations, i.e., to the form

$$|J,M\rangle = \sum_{m=-j}^{j}\sum_{m'=-j'}^{j'} |j,j',m,m'\rangle\langle j,j',m,m'|J,M\rangle\,, \tag{10.64}$$

is known as the Clebsch-Gordan decomposition. The various coefficients that appear are known as Clebsch-Gordan coefficients. For example, the numbers $\frac{1}{\sqrt{6}}$ and $\sqrt{\frac{2}{3}}$ in

$$|2,0\rangle = \frac{1}{\sqrt{6}}\left(|-1,1\rangle + 2|0,0\rangle + |1,-1\rangle\right)\,. \tag{10.65}$$

In other words, $|J,M\rangle$ is a linear combination of $|j,j',m,m'\rangle$ with the Clebsch-Gordan coefficients given by the numbers $\langle j,j',m,m'|J,M\rangle$. Since these vanish unless $m+m'=M$, the double sum in (10.64) reduces to a single sum.

10.3.2 Wigner-Eckert Theorem (Non-examinable)

Remember when we looked at time dependent perturbation theory we saw that transition rates depended on terms of the form

$$\langle n|V|i\rangle \tag{10.66}$$

where V is a perturbation term to the Hamiltonian and $|i\rangle$ and $|n\rangle$ are some eigenstates of the original Hamiltonian. In the context of atomic and molecular physics the perturbation is often invariant under $SO(3)$ and the initial and final states are often initially degenerate angular momentum states, i.e., $|i\rangle = |\alpha,j,m\rangle$ and $|n\rangle = |\alpha',j',m'\rangle$ where α just represents generic other quantum numbers that define the state. In this case, we can use Clebsch-Gordan coefficients to simplify the computation of these terms and the theorem that allows us to do so is called the Wigner-Eckart theorem.

Consider an operator V that transforms under the group $SO(3)$. The Wigner-Eckert theorem states that for a matrix element:

$$\langle\alpha',j',m'|V_{JM}|\alpha,j,m\rangle = \begin{pmatrix} j' & J & j \\ m' & M & -m \end{pmatrix}\langle\alpha',j'||V_J||\alpha,j\rangle, \tag{10.67}$$

where the first term is a Clebsch-Gordan coefficient and the second term, $\langle \alpha', j' \| V_J \| \alpha, j\rangle$, is the reduced matrix element. The theorem indicates that the amplitude factors into a product of two terms: one term encapsulating the group-theoretical properties of the problem (via Clebsch-Gordan coefficients) and the other representing the dynamics, which is independent of m and m', and group theory cannot help us compute.

The selection rules for transitions under $SO(3)$ symmetry emerge naturally from the Clebsch-Gordan coefficients. In particular, the matrix element in (10.67) vanishes unless:

$$\Delta j = j' - j \leq J, \quad \Delta m = m' - m = M \leq J. \tag{10.68}$$

These constraints explain why only specific transitions occur in atomic spectroscopy and why others are forbidden. For a transition where the operator V transforms as a spherical harmonic Y_J^M, the rules imply that:

- $|j' - j| \leq J$ ensures the total angular momentum change aligns with the symmetry of the operator.
- $M = m' - m$ dictates the projection of angular momentum change.

The intensity of an observed transition is proportional to the absolute square of the matrix element:

$$\text{Intensity} \propto |\langle \alpha', j', m' | V_{JM} | \alpha, j, m\rangle|^2, \tag{10.69}$$

with forbidden transitions resulting from violations of the conditions in (10.68). This example therefore highlights the deep role of group theory in determining physically observed phenomena.

I have included this example here to link back to our study of transition rates earlier on in the course and as a taster of material that you will study in more detail in Jean Philippe Bruntut's atomic physics course next term. However, the Wigner Eckert theorem will be un-examinable this year in QP2.

10.4 Other applications of Group Theory and Lie Algebras

To end, I just want to highlight that Lie Groups and Lie Algebras appear all over the place. I've focussed on their application in angular momentum because this should be most familiar given what you've seen before- and is important to understand in a lot of atomic and molecular physics. But let me just name drop a few other applications and give you a few references as to where you can read more about them. (This is, of course, all non-examinable.)

- **Particle Physics.** The most obvious area where you really need to understand Lie Groups and Lie Algebras in Particle Physics. In fact, if you decide to focus on this in your master's you will start with a TPIV devoted to learning Lie Algebra for Particle physics via this textbook.
- **Controlling quantum systems.** You've seen that Lie Algebras are all about figuring out what Hermitian operators generate what unitary representations of a group... this means you can use your understanding of Lie Algebras to figure out how to design Hamiltonians to implement various unitaries on that systems, that is, how to control that system. This is important if you are an theorist/experimentalist trying to build a quantum computer (or other quantum technology). It is also important if you are a quantum software developer trying to design certain quantum algorithms. For an introduction see this tutorial.

- **Machine Learning (Quantum and Classical)** Why is symmetry important in machine learning? This is explained very nicely in this blog post. Consider everyone's favourite example of a machine learning task: classifying images to decide if they include cats of dogs. (If you want a less inane task consider trying to classify whether an images of tumours contain cancerous cells. Or whether images of galaxies contain supernova.)

  There are many different transformations one can perform to an image of a cat that still leave it as a picture of a cat - e.g. you can rotate it or reflect it and you are still left with an image of a cat (Fig. 10.1).

  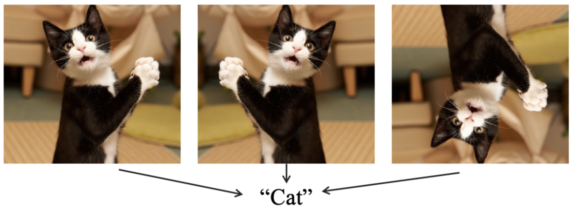

  Figure 10.1: A picture of a rotated cat or flipped cat is still a picture of a cat.

  We want our classifier to be *invariant* under these symmetry transformations. In the context of image processing (or modelling molecules or materials) these symmetry transformations will typically be geometric transformations. Beyond image classification other symmetry transformations, such as permutation invariance, can become important. And, of course, mathematically all these symmetry transformations can represented by the actions of elements of a symmetry group. The theory of Lie Algebras (and group/rep theory more generally) provides us with a way of constructing models with these symmetries in built. For more information on this take a look at my notes from last year, check out this (quite technical) tutorial or this (less technical) tutorial.

- **Classically simulating quantum systems.** As we've discussed before, simulating quantum systems classically is generally hard because it involves multiplying together exponentially large matrices. But if your system as symmetries you can use clever tricks from the theory of Lie Algebras and Lie Groups to make this easier. See this tutorial for more information.

Let $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle \in \mathcal{H}_1$, $|\phi\rangle \in \mathcal{H}_2$, $\dim(\mathcal{H}_1) > \dim(\mathcal{H}_2)$

$\|\psi_1\rangle$	$\|\psi_1\rangle \|\phi\rangle$
$\|\psi_1\rangle \|\psi_2\rangle$	$\|\psi_1\rangle \langle\psi_3\|$

Table 1: Is this loss?

Figure 10.2: And let's end with one more meme. I originally gave this the wooden spoon award because my reaction, similarly to many of you I guess, was 'is this even a meme?'. But having now had it explained to me I have to concede its pretty clever. And if you don't get it - that's just a healthy sign that you don't spend too too much time online.